A GENERAL INVERSION FORMULA FOR SUMMATORY ARITHMETIC FUNCTIONS AND ITS APPLICATION TO THE SUMMATORY FUNCTION OF THE MÖBIUS FUNCTION[1]

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Abstract. We prove an inversion formula for summatory arithmetic functions.
As an application, we obtain an arithmetic relationship between summatory Piltz divisor functions and a sum of the Möbius function over certain integers, denoted by $M(x,y)$. With this relationship, using bounds for the main and remainder terms in the k-divisor problems we deduce conditional and unconditional results concerning $M(x,y)$ and the zero-free region of the Riemann zeta-function and Dirichlet L-functions.

1. Introduction. Let $d_k(n) = \sum_{n_1\cdots n_k=n} 1$, so $d(n) = d_2(n)$ is the number of divisors of n. The Möbius function is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not squarefree}, \\ (-1)^{\omega} & \text{if } n \text{ is squarefree and } n = p_1 \dots p_{\omega}, \end{cases}$$

and the summatory function $M(x)$ (the Mertens function) by

$$M(x) := \sum_{n \leqslant x} \mu(n).$$

Among important problems of analytic number theory are those of investigating the distribution of the summatory functions $\sum_{n\leqslant x} d_k(n)$ and $M(x)$. In the case $k = 2$ the problem for $\sum_{n\leqslant x} d_k(n)$ is called the Dirichlet divisor problem and in the general case it is called the k-divisor problem or the Piltz problem. Both $\sum_{n\leqslant x} d_k(n)$ and $M(x)$ are closely related to the Riemann zeta-function $\zeta(s)$.

For a positive integer n let $P^-(n)$ denote the smallest prime divisor of n and set $P^-(1) = \infty$. For $y \geqslant 2$ and integer $k \geqslant 1$ let

$$d'_k(n,y) = \begin{cases} \sum_{n_1\cdots n_k=n} 1 & \text{if } P^-(n) > y, \\ 0 & \text{else}. \end{cases} \tag{1}$$

Define $d^*_1(n,y) = d'_1(n,y)$ and for integer $k \geqslant 2$

$$d^*_k(n,y) = \begin{cases} \sum\limits_{\substack{n_1\cdots n_k=n \\ n_1,\cdots,n_k \notin \{1,n\}}} 1 & \text{if } P^-(n) > y, \\ 0 & \text{else}. \end{cases}$$

We also define the modified Mertens function

$$M(x,y) := \sum_{\substack{n \leqslant x \\ P^-(n)>y \text{ or } P^-(n/2)>y}} \mu(n).$$

In this article we establish a direct arithmetic relationship between the summatory functions $\sum_{n\leqslant x} d^*_k(n,y)$ and $M(x,y)$ by means of a general inversion formula for summatory functions.

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Theorem 1 (Inversion Formula)**.** *Let*

$$F(X) = \sum_{n \leqslant X} \alpha(n) G\left(\frac{X}{n}\right).$$

Then we have

$$G(X) = \sum_{n \leqslant X} \beta(n) F\left(\frac{X}{n}\right),$$

where

$$\beta(n) = \begin{cases} \frac{1}{\alpha(1)} & \textit{if } n = 1, \\ \frac{1}{\alpha(1)} & \left(-\frac{\alpha(n)}{\alpha(1)} + \sum\limits_{\substack{n_1 n_2 = n \\ n_1, n_2 \notin \{1, n\}}} \frac{\alpha(n_1)\alpha(n_2)}{(-\alpha(1))^2} + \sum\limits_{\substack{n_1 n_2 n_3 = n \\ n_1, n_2, n_3 \notin \{1, n\}}} \frac{\alpha(n_1)\alpha(n_2)\alpha(n_3)}{(-\alpha(1))^3} + \cdots \right) \\ & \textit{if } n > 1. \end{cases} \tag{2}$$

We apply this formula in the following way: Let

$$U(x) = \begin{cases} 1 & \text{if } 1 \leqslant x < 2, \\ 0 & \text{if } x \geqslant 2, \end{cases}$$

$$\alpha(n, y) = \begin{cases} 1 & \text{if } P^-(n) > y, \\ 0 & \text{else.} \end{cases}$$

The identity

$$\frac{1}{\zeta(s)} \prod_{3 \leqslant p \leqslant y} \left(1 - \frac{1}{p^s}\right)^{-1} \zeta(s) \left(1 - \frac{1}{2^s}\right) \prod_{3 \leqslant p \leqslant y} \left(1 - \frac{1}{p^s}\right) = 1 - \frac{1}{2^s}$$

implies the identity for the summatory functions

$$U(X) = \sum_{n \leqslant X} \alpha(n, y) M\left(\frac{X}{n}, y\right),$$

and by the inversion formula (Theorem 1) we get our basic arithmetic identity relating $M(X, y)$ to the summatory divisor functions $\sum_{n \leqslant x} d_k^*(n, y)$:

$$M(X, y) = \sum_{n \leqslant X} \beta(n, y) U\left(\frac{X}{n}\right) = \beta(\lfloor X/2 \rfloor + 1, y) + \beta(\lfloor X/2 \rfloor + 2, y) + \cdots + \beta(\lfloor X \rfloor, y), \tag{3}$$

where, with $\Omega(n)$ standing for the number of prime divisors of n counted according to multiplicity,

$$\beta(n, y) = \sum_{k=1}^{\Omega(n)} (-1)^k d_k^*(n, y).$$

Note that we can take $\frac{\log X}{\log y}$ as the limit of the summation over k, since $d_k^*(n, y) = 0$ if $k > \min\left(\Omega(n), \frac{\log X}{\log y}\right)$.

Let the generating Dirichlet series for the arithmetic function $\beta_l(n,y) = \sum_{k=1}^{l} (-1)^k d_k^*(n,y)$ be denoted by $F_{l,y}(s)$. Since for $d'_k(n,y)$ defined by (1)

$$d_k^*(n,y) = d'_k(n,y) - k d_{k-1}^*(n,y) - \binom{k}{2} d_{k-2}^*(n,y) - \cdots - \binom{k}{k-1} d_1^*(n,y),$$

then $F_{l,y}(s)$ is represented as the linear combination of $\zeta^k(s) \prod_{p \leqslant y} \left(1 - \frac{1}{p^s}\right)^k$ with $1 \leqslant k \leqslant l$. The coefficients can be found from the simultaneous equations:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \binom{2}{1} & 1 & 0 & \dots & \dots & 0 \\ \binom{3}{2} & \binom{3}{1} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{k-1}{k-2} & \binom{k-1}{k-3} & \dots & \binom{k-1}{1} & 1 & 0 \\ \binom{k}{k-1} & \binom{k}{k-2} & \dots & \binom{k}{2} & \binom{k}{1} & 1 \end{pmatrix} \begin{pmatrix} d_1^* \\ d_2^* \\ d_3^* \\ \vdots \\ d_{k-1}^* \\ d_k^* \end{pmatrix} = \begin{pmatrix} d'_1 \\ d'_2 \\ d'_3 \\ \vdots \\ d'_{k-1} \\ d'_k \end{pmatrix}.$$

Let $\Delta_{l,y}(x)$ denote the remainder term in the asymptotic formula

$$\sum_{n \leqslant x} \beta_l(n,y) = \operatorname*{Res}_{s=1} F_{l,y}(s) \frac{x^s}{s} + \Delta_{l,y}(x).$$

Definition 1. *Suppose that $c > 0$, $\alpha_1 > 0$ and α_2 are real, $x \geqslant 2$, $3 \leqslant y(x) \leqslant x$. A number $l_0 = \left\lfloor \frac{\log x}{\log y} \right\rfloor + 1$ is called **the level** for c, α_1, α_2, y, if we have the bounds*

$$\left| \operatorname*{Res}_{s=1} F_{l_0,y}(s) \frac{x^s}{s} \right| \ll x e^{-c(\log x)^{\alpha_1} (\log_{(2)} x)^{\alpha_2}} \tag{4}$$

and

$$|\Delta_{l_0,y}(x)| \ll x e^{-c(\log x)^{\alpha_1} (\log_{(2)} x)^{\alpha_2}}. \tag{5}$$

That is, if l_0 is the level, then (3) *implies that*

$$|M(x,y)| \ll x e^{-c(\log x)^{\alpha_1} (\log_{(2)} x)^{\alpha_2}}.$$

We remark that if we take

$$y = y_0 \asymp \exp\left(c_0 (\log x)^{2/5} (\log_{(2)} x)^{1/5}\right),$$

and if $l_0 = \left\lfloor \frac{\log x}{\log y} \right\rfloor + 1$ were the level for some $c > 0$, $\alpha_1 = 3/5$, $\alpha_2 = -1/5$, then we would have the bound

$$M(x, y_0) = O\left(x e^{-c(\log x)^{3/5} (\log_{(2)} x)^{-1/5}}\right). \tag{6}$$

(The bound (5) can be proved using l_0-th derivative of the generating Dirichlet series as in [Kou12], but the residue term in (4) is quite complicated.) From (6) one could infer, using methods of Pintz [Pi82, Pi84], the best known zero-free region, and then the best known error term in the prime number theorem:

Theorem 2. *With the above assumptions, we have $\zeta(s) \neq 0$ in the region*

$$\sigma > 1 - \frac{c}{(\log(|t|+3))^{2/3} (\log\log(|t|+3))^{1/3}},$$

and, in the prime number theorem,

$$\psi(x) = x + O\left(x e^{-c(\log x)^{3/5} (\log\log x)^{-1/5}}\right) \quad (x \geqslant 3)$$

(*the constant c is not necessarily the same one in every place*).

Remarkably, in the problem of bounding $M_\chi(x,y)$, the sum of the Möbius function twisted by a Dirichlet character, strong estimates for real non-principal characters χ (using $y=y_0$) could be obtained circumventing treatment of the Landau–Siegel zero, since there is no logarithmic derivative in the argument. (And there is no residue term (4) for nonprincipal characters $\chi \bmod q$.)

The above fact allows us to prove the following result on the Landau–Siegel zero:

Theorem 3. *Let χ be a real non-principal primitive character $\bmod q$. Suppose that $\chi(2)=1$ and β_0 is the real zero of Dirichlet L-function $L(s,\chi)$. Then we have*

$$\beta_0 \leqslant 1 - \frac{c}{\log q},$$

where the constant $c>0$ is effectively computable.

This theorem is proved in Section 4. By considering modifications of the functions $M_\chi(x,y)$, namely, those corresponding to the identity

$$\frac{1}{L(s,\chi)}\left(1-\frac{1}{2^s}\right)\prod_{2\leqslant p\leqslant y}\left(1-\frac{\chi(p)}{p^s}\right)^{-1} L(s,\chi)\prod_{2\leqslant p\leqslant y}\left(1-\frac{\chi(p)}{p^s}\right) = 1-\frac{1}{2^s},$$

the theorem can be proved for $\chi(2)=-1$ and hence for all real non-principal primitive characters $\bmod q$.

2. Inversion formula: A simple numerical example. Here we illustrate usage of the inversion formula for a particular choice of the functions F, G and a small value of X.

The Dirichlet convolution of arithmetic functions f, g is defined by $f*g(n)=\sum_{ab=n} f(a)g(b)$.

Lemma 1 (Mertens)**.**

$$\Delta(X) = \psi(X) - \lfloor X \rfloor = \sum_{n\leqslant X} (\log n - d(n)) M\left(\frac{X}{n}\right).$$

Proof.

$$\Delta(X) = \psi(X) - \lfloor X \rfloor = \sum_{m\leqslant X} (\log - d)*\mu(m) = \sum_{m\leqslant X}\sum_{n|m} (\log n - d(n))\mu\left(\frac{m}{n}\right) = \sum_{n\leqslant X} (\log n - d(n)) \sum_{h\leqslant \frac{X}{n}} \mu(h).$$

The following lemma is a consequence of Lemma 1 and the inversion formula (Theorem 1).

Lemma 2. *We have*

$$M(X) = \sum_{n\leqslant X} b_n \Delta\left(\frac{X}{n}\right),$$

where

$$b_n = \begin{cases} -1 & \text{if } n=1,\\ (-1)\Bigg(\log n - d(n) + \sum\limits_{\substack{n_1n_2=n\\ n_1,n_2\notin\{1,n\}}} (\log n_1 - d(n_1))(\log n_2 - d(n_2)) & \\ + \sum\limits_{\substack{n_1n_2n_3=n\\ n_1,n_2,n_3\notin\{1,n\}}} (\log n_1 - d(n_1))(\log n_2 - d(n_2))(\log n_3 - d(n_3)) + \cdots \Bigg) & \text{if } n>1. \end{cases}$$

Let $X = 6$. Then we have:

$$\Delta_{(1)}(X) = \psi(X) - \lfloor X \rfloor = 2\log 2 + \log 3 + \log 5 - 6, \qquad M_{(1)}(X) = M(X) = -1,$$
$$\Delta_{(2)}(X) = \psi\left(\frac{X}{2}\right) - \left\lfloor \frac{X}{2} \right\rfloor = \log 2 + \log 3 - 3, \qquad M_{(2)}(X) = M\left(\frac{X}{2}\right) = -1,$$
$$\Delta_{(3)}(X) = \psi\left(\frac{X}{3}\right) - \left\lfloor \frac{X}{3} \right\rfloor = \log 2 - 2, \qquad M_{(3)}(X) = M\left(\frac{X}{3}\right) = 0,$$
$$\Delta_{(4)}(X) = \psi\left(\frac{X}{4}\right) - \left\lfloor \frac{X}{4} \right\rfloor = -1, \qquad M_{(4)}(X) = M\left(\frac{X}{4}\right) = 1,$$
$$\Delta_{(5)}(X) = \psi\left(\frac{X}{5}\right) - \left\lfloor \frac{X}{5} \right\rfloor = -1, \qquad M_{(5)}(X) = M\left(\frac{X}{5}\right) = 1,$$
$$\Delta_{(\lfloor X \rfloor)}(X) = \psi\left(\frac{X}{\lfloor X \rfloor}\right) - \left\lfloor \frac{X}{\lfloor X \rfloor} \right\rfloor = -1, \qquad M_{(\lfloor X \rfloor)}(X) = M\left(\frac{X}{\lfloor X \rfloor}\right) = 1.$$

By Lemma 1, we can set up the nonsingular system of simultaneous linear equations with the upper triangular sparse matrix:

$$\begin{pmatrix} \Delta_{(1)} \\ \Delta_{(2)} \\ \Delta_{(3)} \\ \Delta_{(4)} \\ \Delta_{(5)} \\ \Delta_{(6)} \end{pmatrix} = \begin{pmatrix} -1 & \log 2 - 2 & \log 3 - 2 & 2\log 2 - 3 & \log 5 - 2 & \log 2 + \log 3 - 4 \\ 0 & -1 & 0 & \log 2 - 2 & 0 & \log 3 - 2 \\ 0 & 0 & -1 & 0 & 0 & \log 2 - 2 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} M_{(1)} \\ M_{(2)} \\ M_{(3)} \\ M_{(4)} \\ M_{(5)} \\ M_{(6)} \end{pmatrix}.$$

Solving the system by Cramer's rule, as in the proof (see below) of the inversion formula (Theorem 1), we obtain

$$M_{(1)}(X) = M(X) =$$
$$\left(-1,\quad 2 - \log 2,\quad 2 - \log 3,\quad -(\log 2)^2 + 2\log 2 - 1,\quad 2 - \log 5,\quad -2\log 2 \log 3 + 3\log 2 + 3\log 3 - 4\right) \begin{pmatrix} \Delta_{(1)} \\ \Delta_{(2)} \\ \Delta_{(3)} \\ \Delta_{(4)} \\ \Delta_{(5)} \\ \Delta_{(6)} \end{pmatrix},$$

which coincides with the assertion of Lemma 2.

3. Proof of Inversion formula.

Theorem 1 (Inversion Formula). *Let*

$$F(X) = \sum_{n \leqslant X} \alpha(n) G\left(\frac{X}{n}\right). \tag{7}$$

Then we have

$$G(X) = \sum_{n \leqslant X} \beta(n) F\left(\frac{X}{n}\right),$$

where

$$\beta(n) = \begin{cases} \frac{1}{\alpha(1)} & \text{if } n = 1, \\ \frac{1}{\alpha(1)} & \left(-\frac{\alpha(n)}{\alpha(1)} + \sum\limits_{\substack{n_1 n_2 = n \\ n_1, n_2 \notin \{1, n\}}} \frac{\alpha(n_1)\alpha(n_2)}{(-\alpha(1))^2} + \sum\limits_{\substack{n_1 n_2 n_3 = n \\ n_1, n_2, n_3 \notin \{1, n\}}} \frac{\alpha(n_1)\alpha(n_2)\alpha(n_3)}{(-\alpha(1))^3} + \cdots \right) \\ & \text{if } n > 1. \end{cases} \tag{8}$$

Proof. We use the following notation:

$$\begin{aligned} F_{(1)}(X) &= F(X), & G_{(1)}(X) &= G(X), \\ F_{(2)}(X) &= F\left(\frac{X}{2}\right), & G_{(2)}(X) &= G\left(\frac{X}{2}\right), \\ &\cdots, & &\cdots, \\ F_{(\lfloor X\rfloor)}(X) &= F\left(\frac{X}{\lfloor X\rfloor}\right), & G_{(\lfloor X\rfloor)}(X) &= G\left(\frac{X}{\lfloor X\rfloor}\right). \end{aligned}$$

By (7), we can set up the nonsingular system of simultaneous linear equations with the upper triangular sparse matrix A:

$$\begin{aligned} F_{(1)}(X) &= \sum_{1\leqslant n\leqslant X} \alpha(n) G_{(n)}(X), \\ F_{(2)}(X) &= \sum_{1\leqslant n\leqslant X/2} \alpha(n) G_{(2n)}(X), \\ &\cdots, \\ F_{(\lfloor X\rfloor)}(X) &= \sum_{1\leqslant n\leqslant X/\lfloor X\rfloor} \alpha(n) G_{(\lfloor X\rfloor n)}(X). \end{aligned}$$

Dividing every equation by $-\alpha(1)$ we obtain the new matrix A' with the new entries $\alpha'(n)$. By Cramer's rule,

$$G_{(1)}(X) = G(X) = \sum_{n\leqslant X} \beta(n) F\left(\frac{X}{n}\right),$$

where

$$\beta(n) = \frac{(-1)^{n+1}\det {A'}_{n,1}}{(-\alpha(1))\det A'}.$$

For $n \geqslant 3$ we multiply i-th row $\overline{r}_i$, starting from $i = 2$, of matrix ${A'}_{n,1}$ by

$$S_i = \left(\alpha'(i) + \sum_{\substack{n_1n_2=i \\ n_1,n_2\notin\{1,i\}}} \alpha'(n_1)\alpha'(n_2) + \sum_{\substack{n_1n_2n_3=i \\ n_1,n_2,n_3\notin\{1,i\}}} \alpha'(n_1)\alpha'(n_2)\alpha'(n_3) + \cdots \right),$$

and make the assignment $\overline{r}_i \leftarrow S_i\overline{r}_i + \overline{r}_{i-1}$ (for $2 \leqslant i \leqslant (n-1)$) to transform ${A'}_{n,1}$ to an upper triangular matrix. This yields formula (8) for $\beta(n)$.

4. Proof of Theorem 3.

Preliminary lemma. The following lemma is a modification of [Kou12, Lemma 2.1].

Lemma 3. *Let $M \geqslant 1$, $c \geqslant 1$, $D \subset \mathbb{C}$ an open set, and $s \in D$. Consider a function $F : D \to \mathbb{C}$ that is differentiable l times at s and its derivatives satisfy the bound $|F^{(j)}(s)| \leqslant j!M^j$ for $1 \leqslant j \leqslant l$, and $|F(s)| \leqslant c$. Then for an integer k, $1 \leqslant k \leqslant l$, we have*

$$\left|\left(F^k(s)\right)^{(l)}\right| \leqslant l!(2c)^k(8M)^l.$$

Proof. We have the identity

$$\left(F^k(s)\right)^{(l)} = l! \sum_{\substack{a_1+2a_2+\cdots=l \\ a_1+a_2+\cdots\leqslant k}} \frac{k!}{(k-a_1-a_2-\cdots)!a_1!a_2!\cdots} F^{k-a_1-a_2-\cdots}(s) \left(\frac{F'(s)}{1!}\right)^{a_1} \left(\frac{F''(s)}{2!}\right)^{a_2} \cdots \tag{9}$$

Using the inequality

$$\frac{(a+b)!}{a!b!} \leqslant 2^{a+b}$$

we obtain

$$\frac{k!}{(k-a_1-a_2-\cdots-a_l)!a_1!a_2!\cdots a_l!} \leqslant 2^k \frac{(a_1+a_2+\cdots+a_l)!}{a_1!a_2!\cdots a_l!}$$

and

$$\begin{aligned}\frac{(a_1+a_2+\cdots+a_l)!}{a_1!a_2!\cdots a_l!} &\leqslant 2^{a_1+a_2+\cdots+a_l}\frac{(a_2+a_3+\cdots+a_l)!}{a_2!a_3!\cdots a_l!}\\ &\leqslant 2^{a_1+2(a_2+\cdots+a_l)}\frac{(a_3+a_4+\cdots+a_l)!}{a_3!a_4!\cdots a_l!}\\ &\vdots\\ &\leqslant 2^{a_1+2a_2+\cdots+la_l}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{a_1+2a_2+\cdots=l}\frac{(a_1+a_2+\cdots+a_l)!}{a_1!a_2!\cdots a_l!} &\leqslant 2^l \sum_{I\subset\{1,\dots,l\}}\sum_{\substack{\sum_{i\in I} ia_i=l\\ a_i\geqslant 1\ (i\in I)}} 1 \leqslant 2^l \sum_{I\subset\{1,\dots,l\}}\prod_{i\in I}\frac{l}{i}\\ &= 2^l\prod_{i=1}^{l}\left(1+\frac{l}{i}\right) = 2^l\binom{2l}{l} \leqslant 8^l.\end{aligned}$$

Now the lemma follows from the obtained inequalities and (9). □

Proof of Theorem 3. Consider the case of χ an even character, that is, $\chi(-1)=1$. Following Pintz [Pi82], define the entire function

$$g_\chi(s) := \frac{L(s-1,\chi)}{(s-1-\beta_0)\prod_{\nu=1}^{2}(s-1+2\nu)},$$

and let

$$\lambda := \log X - 2,$$

$$r_\lambda(H) := \frac{1}{2\pi i}\int_{(3)} e^{s^2/\lambda+Hs} g_\chi(s)\,ds.$$

Using formula

$$\int_1^\infty \frac{M_\chi(x,y_0)}{x^s}dx = \frac{1}{(s-1)L(s-1,\chi)}\prod_{3\leqslant p\leqslant y_0}\left(1-\frac{\chi(p)}{p^{s-1}}\right)^{-1},$$

where $\Re s=\sigma>2$ and $y_0 \asymp \exp\left(c(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$, and interchanging the integrals, we

get

$$U := \int_1^\infty M_\chi(x, y_0) r_\lambda(\lambda - \log x) dx$$
$$= \frac{1}{2\pi i} \int_{(3)} e^{s^2/\lambda + \lambda s} g_\chi(s) \int_1^\infty \frac{M_\chi(x, y_0)}{x^s} dx\, ds$$
$$= \frac{1}{2\pi i} \int_{(3)} e^{s^2/\lambda + \lambda s} \frac{\prod_{3 \leqslant p \leqslant y_0} \left(1 - \frac{\chi(p)}{p^{s-1}}\right)^{-1}}{(s - 1 - \beta_0)(s-1) \prod_{\nu=1}^2 (s - 1 + 2\nu)} ds.$$

Thus,

$$U = \operatorname*{Res}_{s=1+\beta_0} e^{s^2/\lambda + \lambda s} \frac{\prod_{3 \leqslant p \leqslant y_0} \left(1 - \frac{\chi(p)}{p^{s-1}}\right)^{-1}}{(s - 1 - \beta_0)(s-1) \prod_{\nu=1}^2 (s - 1 + 2\nu)} + O\left(X^{1+\beta_0 - 1/\log y_0} (\log y_0)^c\right)$$
$$= \frac{e^{(1+\beta_0)^2/\lambda + (1+\beta_0)\lambda} \prod_{3 \leqslant p \leqslant y_0} \left(1 - \frac{\chi(p)}{p^{\beta_0}}\right)^{-1}}{\beta_0 \prod_{\nu=1}^2 (\beta_0 + 2\nu)} + O\left(X^{1+\beta_0 - 1/\log y_0} (\log y_0)^c\right)$$
$$\gg \frac{X^{1+\beta_0}}{(\log y_0)^c}.$$

Now we derive an upper bound for $|U|$. First, if $H \leqslant -2$ ($H = \lambda - \log x = \log X - \log x - 2$), we integrate along the line $\sigma = \lambda$ instead of $\sigma = 3$:

$$|r_\lambda(H)| \leqslant \frac{1}{2\pi} \int_{-\infty}^\infty e^{\lambda - t^2/\lambda - |H|\lambda} \frac{2}{\lambda - 2} dt \leqslant e^{-\lambda(|H|-1)},$$

and by the trivial estimate $|M_\chi(x, y_0)| \leqslant x$ we have for the part of the integral with $x \geqslant X$:

$$\left| \int_X^\infty M_\chi(x, y_0) r_\lambda(\lambda - \log x) dx \right| \leqslant \int_{e^{\lambda+2}}^\infty x e^{-\lambda(\log x - \lambda - 1)} dx$$
$$= \frac{e^{\lambda^2 + \lambda} x^{2-\lambda}}{2 - \lambda} \Bigg|_{x = e^{\lambda+2}}^{x=+\infty} = \frac{e^{\lambda+4}}{\lambda - 2} \ll X.$$

Next, we move the line of integration in $r_\lambda(H)$ to $\sigma = 0$. By the functional equation for Dirichlet L-functions and the Stirling formula we get

$$L(-1 + it, \chi) \ll q^{3/2} |t|^{3/2},$$

hence for an arbitrary H

$$|r_\lambda(H)| \ll \int_{-\infty}^{+\infty} \frac{q^{3/2}|t|^{3/2}}{|t|^3} dt \ll q^{3/2},$$

so

$$|U| \ll q^{3/2} \int_1^X |M_\chi(x, y_0)|\, dx + O(X).$$

Combining the upper and the lower estimates, we obtain

$$\frac{X^{1+\beta_0}}{q^{3/2}(\log y_0)^c} \ll \int\limits_1^X |M_\chi(x, y_0)|\, dx + O\left(\frac{X}{q^{3/2}}\right). \tag{10}$$

To derive an upper bound for $|M_\chi(x, y_0)|$ we use l_0-th derivative of the function $F_{l_0,y_0}(s,\chi)$ and an upper bound for the summatory function

$$\sum_{n \leqslant x} \beta_{l_0,\chi}(n, y_0)(-\log n)^{l_0}$$

with $y_0 \asymp \exp\left(c_0(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$ and $l_0 = \left\lfloor \frac{\log X}{\log y_0} \right\rfloor + 1$ (see discussion before Definition 1). By an argument similar to that of Koukoulopoulos [Kou12, Lemma 4.1 and Section 6], denoting by c a certain positive constant, not necessarily the same one in every place, for $y_0 \asymp \exp\left(c_0(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$, $l_0 = \left\lfloor \frac{\log X}{\log y_0} \right\rfloor + 1$, $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$,

$$V_t = \exp\left((\log(3+|t|))^{2/3}(\log\log(3+|t|))^{1/3}\right),$$

$q \asymp \exp\left(c(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$, and χ a nonprincipal Dirichlet character mod q we have that

$$\left|L_{y_0}^{(j)}(s,\chi)\right| = \left|\left(L(s,\chi)\prod_{p \leqslant y_0}\left(1 - \frac{\chi(p)}{p^s}\right)\right)^{(j)}\right| \ll \frac{j!(c\log(y_0 q V_t))^{j+1}}{\log y_0},$$

and, using Lemma 3, for every $y \geqslant 2$

$$\begin{aligned}
\sum_{n \leqslant y} \beta_{l_0,\chi}(n, y_0)(\log n)^{l_0} \log\frac{y}{n} &= \frac{(-1)^{l_0}}{2\pi i}\int_{\Re(s)=1+\frac{1}{\log y}} \left(F_{l_0,y_0}(s,\chi)\right)^{(l_0)} \frac{y^s}{s^2} ds \\
&\ll y\int_{-\infty}^{+\infty} (c_1 l_0 \log(y_0 q V_t))^{l_0} \frac{dt}{1+t^2} \\
&\ll y(c_2 l_0^5 \log l_0)^{l_0/3}
\end{aligned} \tag{11}$$

for some $c_1, c_2 \geqslant 1$. For $X^c \ll x \ll X$ set

$$\Delta(x) = x\left(\frac{c_2 l_0^{5/3}(\log l_0)^{1/3}}{\log x}\right)^{l_0/2}$$

and note that $\Delta(x) \geqslant \sqrt{x}$, since $x \geqslant (\log x)^{l_0}/{l_0}^{l_0}$. We assert that

$$\sum_{1 < n \leqslant x} \beta_{l_0,\chi}(n, y_0)(\log n)^{l_0} \ll \Delta(x)(\log x)^{l_0+1}. \tag{12}$$

If $\Delta(x) > x$, then (12) is trivial. (We use the analog of (3) for the Dirichlet L-function and the trivial estimate $|M_\chi(x, y_0)| \leqslant x$.) So assume that $\Delta(x) < x$ and hence the ratio in $\Delta(x)$ is < 1. As in [Kou12], using (11) with $y = x$ and $y = x + \Delta(x)$ and subtracting, we arrive at assertion (12). From (12) by partial summation it follows that

$$\sum_{1 < n \leqslant x} \beta_{l_0,\chi}(n, y_0) \ll 2^{l_0}\Delta(x)\log x \quad (x \geqslant 3). \tag{13}$$

Because we have chosen $l_0 \asymp (\log X)^{3/5}(\log\log X)^{-1/5}$, and $X^c \ll x \ll X$ in (13), we obtain that

$$\sum_{n \leqslant x} \beta_{l_0,\chi}(n, y_0) \ll x e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} \quad (x \geqslant 3)$$

and by the inversion formula, for $X^c \ll x \ll X$ we have

$$|M_\chi(x, y_0)| \ll x e^{-c(\log x)^{3/5}(\log_{(2)} x)^{-1/5}}.$$

Thus, (10) (with $q \asymp \exp\left(c(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$ and $y_0 \asymp \exp\left(c_0(\log X)^{2/5}(\log_{(2)} X)^{1/5}\right)$) implies that

$$\beta_0 \leqslant 1 - \frac{c}{\log q}$$

and the constant $c > 0$ is effectively computable.

References

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